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2008 J. Phys. A: Math. Theor. 41 215401

(http://iopscience.iop.org/1751-8121/41/21/215401)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 215401 (6pp) doi:10.1088/1751-8113/41/21/215401

Noncommutative radial waves

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Received 28 January 2008
Published 7 May 2008
Online at stacks.iop.org/JPhysA/41/215401

Abstract
We study radial waves in a (2+1)-dimensional noncommutative scalar field theory, using operatorial methods. The waves propagate along a discrete radial coordinate and are described by finite series deformations of Bessel-type functions. At radius much larger than the noncommutativity scale $\sqrt{\theta}$, one recovers the usual commutative behaviour. At small distances, classical divergences are smoothed out by noncommutativity.

PACS numbers: 11.10.Nx, 02.70.Bf

Field theories defined over a noncommutative space [1] are interesting, nonlocal but most probably consistent, deformations of the usual ones. They also arise as a particular low-energy limit of string theory in a B-field [2, 3]. Noncommutative (NC) field theories display an intriguing IR/UV mixing (see [4] and later works), demonstrated perturbatively but expected to hold in general. At the classical level, they possess solitonic solutions [5] which have no obvious counterpart in local field theory. Other nontrivial solutions of the equations of motion were also found, cf for instance [6]. However, in spite of considerable progress, NC field theories are far from being well understood, even classically. One interesting issue, which might shed further light on these theories, is the description of oscillation and propagation processes on fuzzy spaces. An example will be discussed here.

The aim of this paper is to describe radial waves in a $(2+1)$-dimensional free scalar field theory with NC spatial coordinates (theories with NC time were claimed to be non-unitary [7]). In contrast to plane waves, radial waves are affected by the presence of noncommutativity. They propagate on a discrete space, provided by the eigenvalues of the radius square operator. Their amplitude solves a discrete wave equation and is given by a finite series, reminiscent of Bessel-type functions. In the large-radius limit (analogous to the 'large quantum number limit' in quantum mechanics) the number of terms of the series grows indefinitely. Then, NC radial waves behave like the usual commutative ones, being described by the asymptotics of cylindrical functions. At small radius, the solutions of the wave equation are nonsingular, as

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they are everywhere, even in the presence of sources. Thus NC field theories may offer a solution to the old problem of *classical* divergences, although they do not seem to qualify at the quantum level.

We will use operatorial methods, which are quite straightforward in the present context. The Weyl–Moyal approach would require an unambiguous way to switch between Cartesian and polar NC coordinates.

The set-up

Let us start with the following action, written in operatorial form,

$$S = \int \mathrm{d}t\, \mathrm{Tr}_{\mathcal{H}} \left(\frac{1}{2}(\dot{\Phi}^2 + \frac{1}{2}[X_i, \Phi]^2 \right), \qquad i = 1, 2. \tag{1}$$

The scalar field Φ is a time-dependent operator acting on the Hilbert space $\mathcal{H}$ on which the algebra

$$[x_1, x_2] = \mathrm{i}\theta \tag{2}$$

is represented. There is no potential term, $V(\Phi) = 0$, since we study free waves. X_i is given by $X_i = p_i + A_i$, where $p_i = \theta^{-1}\epsilon_{ij}x^j$. In the following the gauge field A_i is taken to be zero; consequently we dropped further parts of the action which depend on it. The equations of motion for the field Φ are

$$\ddot{\Phi} + [X_i, [X_i, \Phi]] = \ddot{\Phi} + \frac{1}{\theta^2}[x_i, [x_i, \Phi]] = 0. \tag{3}$$

In Cartesian coordinates, the solution of (3) is straightforward,

$$\Phi \sim \mathrm{e}^{\mathrm{i}(k_1x_1+k_2x_2)-\mathrm{i}\omega t}, \qquad k_1^2 + k_2^2 = \omega^2, \tag{4}$$

and describes plane waves, which are formally identical to the commutative ones. (However (4) has in fact bilocal character, see, e.g., [8].)

The novelty appears when one considers polar coordinates. If one chooses the oscillator basis $\{|n\rangle\}$ given by

$$N|n\rangle = n|n\rangle, \qquad N = \bar{a}a, \qquad a = \frac{1}{\sqrt{2\theta}}(x_1 + \mathrm{i}x_2), \tag{5}$$

the equations of motion become

$$\ddot{\Phi} + \frac{2}{\theta}[a, [\bar{a}, \Phi]] = 0. \tag{6}$$

$N = \frac{1}{2}\left(\frac{x_1^2+x_2^2}{\theta} - 1\right)$ is basically the radius square operator in units of θ. Thus, radial symmetry amounts to the assumption $\Phi = \Phi(N)$. Then Φ is diagonal in the $|n\rangle$ basis, and its components are time-dependent c-numbers, $\langle n|\Phi(t)|n\rangle \equiv \Phi_n(t)$. They satisfy the equation

$$\ddot{\Phi}_n - \frac{2}{\theta}(n\Delta^2\Phi_{n-1} + \Delta\Phi_n) = 0, \qquad n = 0, 1, 2, \ldots \tag{7}$$

in which the discrete derivative operator Δ is defined by

$$\Delta(\Phi_n) = \Phi_{n+1} - \Phi_n. \tag{8}$$

(Alternatively, one could have diagonalized Φ in the $|n\rangle$ basis, without assuming radial symmetry. This can be done only once however, thus only one particular solution of (6) would be diagonal. The other would be bilocal, i.e. of the form $\langle n'|\Phi(t)|n\rangle$, with $n' \neq n$.)

If one assumes the time dependence of Φ_n to be of the form $\mathrm{e}^{\mathrm{i}\omega t}$, one gets the difference equation

$$n\Delta^2\Phi_{n-1} + \Delta\Phi_n + \lambda\Phi_n = 0, \qquad n = 0, 1, 2, \ldots \tag{9}$$

where $\lambda = \theta\omega^2/2$ for a massless scalar field, $2\lambda/\theta = \omega^2 - m^2$ for a massive field and $2\lambda/\theta = \omega^2 + m^2$ for a tachyon.

Solution of the equation of motion

Equation (9) describes travelling or stationary waves on a semi-infinite discrete space, namely the points $n = 0, 1, 2, \ldots$. We will find its two independent linear solutions in the form of (eventually finite) power series. To do so, note that the standard power $n^k = n \cdot n \cdot \ldots \cdot n$ does not behave simply under the action of Δ. To adapt the usual logic of power series solutions to the above discrete equation, we define a different type of 'power of n' (n is still a positive integer):

$$n^{(k)} \equiv n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}. \tag{10}$$

It has a quite useful property that $\Delta n^{(k)} = kn^{(k-1)}$. The definition (10) can be extended to arbitrary real powers. Having in mind the Gamma function representation of the factorial, $n! = \Gamma(n) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} t^n$ [for simplicity in notation we write $\Gamma(n)$ instead of the usual $\Gamma(n+1)$], we define for any real k

$$n^{(k)} \equiv \frac{\Gamma(n)}{\Gamma(n-k)}. \tag{11}$$

We now search for a solution $\Phi(n, \sigma)$ of the form

$$\Phi(n,\sigma) = \sum_{k=0}^{\infty} a_k(\sigma) n^{(k+\sigma)}, \tag{12}$$

with σ an arbitrary parameter, to be fixed by the equation. Substituting (12) into (9), one obtains a recurrence relation for the coefficients $a_k(\sigma)$,

$$a_k(\sigma) = \frac{(-\lambda)}{(k+\sigma)^2} a_{k-1}(\sigma) = \frac{(-\lambda)^k}{(k+\sigma)^2(k-1+\sigma)^2\cdots(1+\sigma)^2} a_0 \tag{13}$$

and a quadratic condition for σ,

$$\sigma^2 = 0. \tag{14}$$

Thus a first solution of our equation is

$$\Phi_1(n) = a_0 \sum_{k=0}^{n} \frac{(-\lambda)^k}{(k!)^2} n^{(k)}. \tag{15}$$

It is given by a finite sum, since $n^{(p)} = 0$ for $n > p$, n, p are positive integers. It is understood that, if one calculates a discrete derivative of (i.e. apply Δ to) the above solution, one should put $\sigma = 0$ only after operating with Δ. a_0 is a dimensionful constant, which will be dropped from now on; it can be reinstated at any moment.

Since equation (14) has two equal roots, the above procedure provides only one solution of (9). Adapting again the methods used for continuous variables (see, for instance, [9]) to the discrete case, we search a second linearly-independent solution of (9) of the form

$$\Phi_2(n) = \left[\frac{\partial \Phi(n,\sigma)}{\partial \sigma}\right]_{\sigma=0}. \tag{16}$$

In the above equation, $\Phi(n, \sigma)$ has the form (12), with $a_k(\sigma)$ given by (13). In order to actually evaluate $\Phi_2(n)$, we need to take the derivatives with respect to σ of $a_k(\sigma)$ and of $n^{(k+\sigma)}$, at $\sigma = 0$. It is easy to show, using (13), that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\sigma} a_k(\sigma)\right|_{\sigma=0} = -2a_k(0) H_k, \tag{17}$$

where H_k is (up to a constant) a discrete version of the logarithmic function,

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}, \tag{18}$$

with $H_0 = 0$ however. To find $\frac{\mathrm{d}}{\mathrm{d}\sigma} n^{(k+\sigma)}$, we use the extended definition (11), $n^{(k+\sigma)} = \frac{\Gamma(n)}{\Gamma(n-k-\sigma)}$ and $\frac{\mathrm{d}}{\mathrm{d}\sigma}\Gamma(\sigma) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} t^\sigma \log t$. This leads to

$$\left[\frac{\mathrm{d}n^{(k+\sigma)}}{\mathrm{d}\sigma}\right]_{\sigma=0} = \frac{n^{(k)}}{(n-k)!}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} t^{n-k} \log t \equiv \frac{n^{(k)}}{(n-k)!} I_{n-k}. \tag{19}$$

By successive integration by parts, one obtains $\frac{I_{n-k}}{(n-k)!} = H_{n-k} + I_0$. I_0 can be dropped, since it leads to a term proportional to the already known $\Phi_1(n)$. In the end one obtains

$$\Phi_2(n) = \sum_{k=0}^{n} \frac{(-\lambda)^k}{(k!)^2} n^{(k)} (H_{n-k} - 2H_k). \tag{20}$$

By direct calculation one straightforwardly shows that Φ_2 obeys equation (9), in the same fashion in which Φ_1 does. The last thing to be shown is the linear independence of the two solutions, which amounts [10] to proving that

$$W(n) \equiv \Phi_1(n+1)\Phi_2(n) - \Phi_1(n)\Phi_2(n+1) \neq 0, \qquad \forall\, n \geqslant 0. \tag{21}$$

Equation (21) is proved by direct calculation, for instance by restricting to the (eventually nonzero) coefficient of the highest power of λ appearing in $W(n)$. The general solution of (9) is thus a linear combination of $\Phi_1(n)$ and $\Phi_2(n)$,

$$\Phi(n) = c_1\Phi_1(n) + c_2\Phi_2(n), \tag{22}$$

with the coefficients $c_{1,2}$ fixed by some physical boundary conditions.

Large distances: commutative limit

We now consider the $n \to \infty$ limit in order to see how the precedent solutions behave at distances $r \gg \sqrt{\theta}$ from the $n = 0$ point. (This can also be seen as a small θ limit.) Using $\lambda = \theta\omega^2/2$ and $n = \frac{r^2}{2\theta} \to \infty$, $\Phi_1(n)$ becomes, as a function of r, the zero-order Bessel function of first type:

$$\Phi_1(n) \overset{n\to\infty}{\to} f_1(r) = \sum_{k=0}^{\infty} \frac{(-1)^k(\omega r)^{2k}}{(k!)^2 2^{2k}} = J_0(\omega r) \overset{r\to\infty}{\sim} \sqrt{\frac{2}{\pi\omega r}}\cos(\omega r - \pi/4). \tag{23}$$

We see that $f_1(r)$ is independent of θ. This is not the case for the function of r one would obtain at finite n, which diverges as $\theta \to 0$. We will encounter only Bessel functions of zero order in what follows, since the angular dependence of Φ is lost, due to (2). We stress that the absence of radial symmetry would not introduce angular dependence in the solutions of the equations of motion. It would lead to nonzero (bilocal) solutions of the type $\langle n'|\Phi|n\rangle$, $n' \neq n$.

Similarly, Φ_2 becomes

$$\Phi_2(n) \to f_2(r) = \sum_{k=0}^{\infty} \frac{(-1)^k(\omega r)^{2k}}{(k!)^2 2^{2k}} [2\ln(\omega r) - 2H_k + \gamma - \ln(2\theta\omega^2)]. \tag{24}$$

γ is the Euler–Mascheroni constant, $\gamma = \lim_{k=\infty}(H_k - \ln k) \simeq 0.5772$. Thus $f_2(r)$ still depends on θ, via a logarithmic term, which renders the $\theta \to 0$ limit singular. Using the series expansion of the Bessel function of second kind (Neumann function) [9],

$Y_0(\omega r) \overset{r\to\infty}{\sim} \sqrt{\frac{2}{\pi\omega r}}\sin(\omega r - \pi/4),$

$$Y_0(\omega r) = \frac{2}{\pi}\left(\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(\omega r)^{2k}}{(k!)^2 2^{2k}}H_k + (\gamma + \ln(\omega r/2))J_0(\omega r)\right), \tag{25}$$

one sees that $f_2(r)$ is a linear combination of Bessel functions of first and second kinds: $f_2(r)/\pi = Y_0(\omega r) + (\gamma + \ln(2\theta\omega^2))J_0(\omega r)$.

Hence, the $n \to \infty$ limits of the solutions $\Phi_1(n)$ and $\Phi_2(n)$ obey the Bessel equation. This is in agreement with the $n = \frac{r^2}{2\theta} \to \infty$ limit of the difference operator entering equation (9),

$$\frac{2}{\theta}(n\Delta^2\Phi_{n-1} + \Delta\Phi_n) \overset{n\to\infty}{\to} \frac{2}{\theta}\left(n\frac{\mathrm{d}^2}{\mathrm{d}n^2} + \frac{\mathrm{d}}{\mathrm{d}n}\right)\Phi(n) \overset{n=\frac{r^2}{2\theta}}{=} \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)f(r). \tag{26}$$

Thus, at $r \gg \sqrt{\theta}$, NC radial waves behave like ordinary, commutative ones. This allows us to find, via a 'correspondence principle', which combinations of $\Phi_1(n)$ and $\Phi_2(n)$ correspond to stationary, respectively travelling NC waves. In the commutative case, standing waves are described by the $J_0(r)$ function, whereas radially expanding ones by the first Hankel function $H_0^{(1)}(r) = J_0(r) + \mathrm{i}Y_0(r)$. Hence, the linear combination of $\Phi_1(n)$ and $\Phi_2(n)$ which at $n \to \infty$ tends to $J_0(\omega r)$ will describe standing noncommutative waves. This is obviously $\Phi_1(n)$, which consequently solves finite-area-boundary-value problems with radial symmetry, describing standing oscillations. On the other hand, the function which tends to $H_0(\omega r)$ as $r \to \infty$, namely

$$\Phi_3(n) = \Phi_1(n) + \frac{\mathrm{i}}{\pi}(\Phi_2(n) + [\gamma + \ln(\theta\omega^2/2)]\Phi_1(n)), \tag{27}$$

represents a radial NC wave propagating outwards, towards $n = \infty$. Any solution $\Phi(n)$ of (9) can be written as a linear superposition of $\Phi_1(n)$ and either $\Phi_2(n)$ or $\Phi_3(n)$, with coefficients determined by the boundary conditions one wishes to impose. It is understood that all the above solutions are multiplied by a dimensionful, otherwise arbitrary, constant; the same will apply for sources.

Small distance: no classical divergences

It is worth noting that, in sharp contrast to the commutative case, in which Hankel and Neumann functions are singular at the origin, the functions $\Phi_{2,3}$ are nowhere singular (except when $\theta = 0$). This suggests that, although not finite in quantum perturbation theory, fields defined over noncommutative spaces may not display classical divergences. This happens simply because the sources are not localized (also, one has no access to the origin: $r/\sqrt{\theta} = \sqrt{2n+1} \geqslant 1$). In order to rigorously support such a claim, one has to include sources in the calculation, by solving the inhomogeneous version of equation (9),

$$(n+1)\Phi(n+1) - (2n+1-\lambda)\Phi(n) + n\Phi(n-1) = j(n). \tag{28}$$

We present below a simple way to do this, and demonstrate the non-divergent character of the solutions. Consider first a nonzero source $j(n_0)$ located at the point n_0. The general solution of (28) is then the sum of the already found homogeneous solution (22) and of a particular solution Φ_p, to be found from (28) when $j(n) = j(n_0)\delta_{n,n_0}$. To find Φ_p we adapt the method of variation of constants to the discrete case [10], and search for

$$\Phi_p(n) = c_1(n)\Phi_1(n) + c_2(n)\Phi_2(n). \tag{29}$$

Assuming that $c_{1,2}(n)$ is constant except for a jump at the source location n_0,

$$c_i(n+1) - c_i(n) = f_1(n)\delta_{n_0,n}, \qquad i = 1, 2, \tag{30}$$

and using the fact that $\Phi_{1,2}(n)$ solve the homogeneous equation, we obtain

$$f_1(n) = \frac{\Phi_2(n)}{(n+1)W(n)}, \qquad f_2(n) = -\frac{\Phi_1(n)}{(n+1)W(n)}, \qquad \forall\, n \geqslant 0. \tag{31}$$

$W(n)$ is the discrete Wronskian defined in equation (21), which is nonzero due to the linear independence of Φ_1 and Φ_2. Although in the physically most interesting case $n_0 = 0$ the difference equation (28) becomes first order, the above method works the same due to the simple Ansatz (30).

The solution for an arbitrary distribution of charges $j(n), \forall\, n$, is now obtained by linear superposition of the above type of solutions. It does not display singularities.

Let us conclude with a summary of what we have shown:

- On the NC plane defined by $[x^1, x^2] = \mathrm{i}\theta$, radial waves propagate on a discrete space, given by the eigenvalues $r = \sqrt{2n+1}, n = 0, 1, 2, \ldots$ of the radius square operator. One has no access to the origin ($r = 0$), as one would expect. The amplitude of the waves is given by a finite series, whose number of terms depends on the location at which the field amplitude is calculated: at radius $r = \sqrt{2n+1}\sqrt{\theta}$, one has $n+1$ terms in the series.
- In the large radius limit, $r \gg \sqrt{\theta}$ or $n \to \infty$, the amplitudes become Bessel-type functions, consequently the waves behave like commutative ones.
- At small radius, if $\theta \neq 0$, there are no signs of singularities appearing. This drastic improvement in the behaviour of classical noncommutative theories deserves to be further explored.

Acknowledgments

This work was supported by a MURST (Italy) fellowship. I thank the HEP theory group of the University of Crete for their hospitality, and Kostas Anagnostopoulos, Elias Kiritsis and Corneliu Sochichiu for useful discussions.

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